Name:______________________________

Math 3113 Section 01 *Practice Final Exam* *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit!

1. Let A be an $n \times n$ matrix. Suppose that A has the following property: $A^2 = A$. We call this property, idempotent. Show that A^t is idempotent. Is it true that if both A and B are idempotent, then so it $A + B$?

2. Let A be an $n \times n$ matrix. Suppose that A has the following property: $A^k = 0$ for some $k \geq 1$. We call this property, nilpotent. Show that $I_n - A$ is an invertible matrix. (HINT: The inverse can be explicitly computed, think of factoring.)

3. Let

$$S = \{-t^2 + t + 2, 2t^2 + 2t + 3, 4t^2 - 1\}$$

Determine if S is a basis for P_2, the vector space of all polynomials of degree 2 or less.

4. Let

$$S = \{t^2 + 1, 3t^2 + 2t + 1, 6t^2 + 6t + 3\}$$

Determine if S is a basis for P_2, the vector space of all polynomials of degree 2 or less.

5. Let $T : P_5 \to P_5$ be defined as

$$T(p) = \frac{dp}{dt}$$

Show that T is a linear transformation. Given the basis $B = \{1, t, t^2, t^3, t^4, t^5\}$, find a matrix A that represents T and compute the kernel of T.

6. Let $T : P_3 \to P_4$ be defined as

$$T(p) = \int_0^t p(\tau)\, d\tau$$

Show that T is a linear transformation. Given the basis $B_1 = \{1, t, t^2, t^3\}$ for P_3 and basis $B_2 = \{1, t, t^2, t^3, t^4\}$ for P_4, find a matrix A that represents T and compute the kernel of T.

7. Compute the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix}$$

8. Compute the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{pmatrix}$$

9. Use the Gram-Schmidt process to find an orthonormal basis for $S = \{1, t, t^2\}$ the standard basis for P_2 with the following inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\ dt$$

.

10. Use the Gram-Schmidt process to find an orthonormal basis for $S = \{1, t\}$ the standard basis for P_1 with the following inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\ dt$$

.

11. Find the *nth* order Fourier approximation to the function $f(t) = |t|$ on the interval $[-\pi, \pi]$. Use this result to deduce the famous identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

12. Find the nth order Fourier approximation to the function $f(t) = t^2$ on the interval $[0, 2\pi]$.

13. Let A be an $n \times n$ matrix who has eigenvalues $\lambda_1, \ldots, \lambda_n$ that are all nonnegative. Show

$$\sqrt[n]{\det(A)} \leq \frac{1}{n}\text{tr}(A)$$

(HINT: You will need to use the Cauchy-Schwarz inequality to deduce another helper inequality)